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TESTING HYPOTHESES IN UNBALANCED VARIANCE COMPONENTS MODELS FOR COMPLETE TWO-WAY LAYOUTS

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C O N T E N T S

1. Introduction and summary

Graybill and Hultquist (1961) describe a variance components model as follows: An $(n\times 1)$ vector of observations $\underset{\sim}{Y}$ is assumed to be a linear sum of k+2 quantities,

$$(1.1)\qquad \underset{\sim}{Y} = \underset{\sim}{J}_n\beta_0 + \sum_{i=1}^{k} \underset{\sim}{B}_i\underset{\sim}{\beta}_i + \underset{\sim}{\beta}_{k+1}$$

Here β_0 is a fixed unknown constant. $\underset{\sim}{\beta}_i$ is a $(p_i\times 1)$ vector of multinormally distributed random variables with mean $\underset{\sim}{0}$ and covariance matrix $\sigma_i^2\,\underset{\sim}{I}_{p_i}$. ($\underset{\sim}{I}_k$ denotes a k-dimensional identity matrix and $\underset{\sim}{0}$ a null matrix).
The vectors $\underset{\sim}{\beta}_1,\underset{\sim}{\beta}_2,\ldots,\underset{\sim}{\beta}_{k\times 1}$ are stochastically independent. $\underset{\sim}{J}_k$ is a $(k\times 1)$ vector with all elements equal to 1. $\underset{\sim}{B}_i$ $(i = 1,2,\ldots,k)$ a $(n\times p_i)$ matrix of known constants.

Some general theorems concerning this model have been derived by Graybill and Hultquist (1961) under one or both of the Following assumptions

(i) $\underset{\sim}{A}_i$ and $\underset{\sim}{A}_j$ commute, where $\underset{\sim}{A}_i = \underset{\sim}{B}_i\,\underset{\sim}{B}_i'$ $\quad (i = 1,2,\ldots,k)$

(ii) The matrix $\underset{\sim}{B}_i$ is such that $\underset{\sim}{J}_n'\underset{\sim}{B}_i = r_i\underset{\sim}{J}_{r_i}'$ and $\underset{\sim}{B}_i \cdot \underset{\sim}{J}_{p_i} = \underset{\sim}{J}_n$,

where r_i is a positive integer.

The assumptions (i) are not satisfied in unbalanced models.

In this paper we will consider a special case of model (1.1) without assumption (i), viz. the common variance components model for a complete two-way layout. Spjøtvoll (1968) has treated the same model in a different manner.

In sections 2 and 3 we shall transform our model to a "semi-canonical" form and find a method for obtaining confidence intervals and testing hypotheses concerning the σ_i^2. In section 4 these tests are compared with the corresponding tests in a fixed effects model. In section 5 the test statistics are expressed in terms of the original observations.

2. Modification of the model of Graybill and Hullquist

We consider the following model:

$$(2.1)\qquad y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk};$$

$i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$, and $k = 1,2,\ldots,n_{ij}$. Here μ is a constant, while α_i, β_j, γ_{ij}, and e_{ijk} are independent normally distributed random variables with means 0 and variances σ_A^2, σ_B^2, σ_{AB}^2, and σ^2, respectively.

Define $\bar{y}_{ij} = (1/n_{ij}) \sum_{k=1}^{n_{ij}} y_{ijk}$; $i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$. Then

$$\bar{y}_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \bar{e}_{ij}. \tag{2.2}$$

With $\bar{e}_{ij} = (1/n_{ij}) \sum_{k=1}^{n_{ij}} e_{ijk}$.

For any set of variables a_{ij} $(i = 1,2,\ldots,r;\ j = 1,2,\ldots,s)$, let $\underset{\sim}{a}$ be the vector $(a_{11}, a_{12},\ldots, a_{1s}, a_{21},\ldots,a_{rs})'$. Then $\bar{\underset{\sim}{e}}$ is multivariate normally distributed with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}(\bar{\underset{\sim}{e}}) = \underset{\sim}{K}\sigma^2$, where

$$\underset{\sim}{K} = \text{Diag}\,(n_{11}^{-1}, n_{12}^{-1},\ldots, n_{rs}^{-1}). \tag{2.3}$$

Formula (2.2) may be written in matrix form as

$$\bar{\underset{\sim}{y}} = \underset{\sim}{J}_{rs}\,\mu + \underset{\sim}{B}_1\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_r\end{bmatrix} + \underset{\sim}{B}_2\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_s\end{bmatrix} + \underset{\sim}{B}_3\underset{\sim}{\gamma} + \bar{\underset{\sim}{e}}, \tag{2.4}$$

$$\text{with } \underset{\sim}{B}_1 = \begin{bmatrix}\underset{\sim}{J}_s, & \underset{\sim}{0}, & \ldots, & \underset{\sim}{0}\\ \underset{\sim}{0}, & \underset{\sim}{J}_s, & \ldots, & \underset{\sim}{0}\\ - & - & - & -\\ \underset{\sim}{0}, & \underset{\sim}{0}, & \ldots, & \underset{\sim}{J}_s\end{bmatrix}, \quad \underset{\sim}{B}_2 = \begin{bmatrix}\underset{\sim}{I}_s\\ \underset{\sim}{I}_s\\ \vdots\\ \underset{\sim}{I}_s\end{bmatrix},$$

and $\underset{\sim}{B}_3 = \underset{\sim}{I}_{rs}$, which is of the same form as (1.1). The covariance matrix for $\bar{\underset{\sim}{y}}$ turns out as

$$\underset{\sim}{\Sigma}(\bar{\underset{\sim}{y}}) = \underset{\sim}{B}_1\underset{\sim}{B}_1'\,\sigma_A^2 + \underset{\sim}{B}_2\underset{\sim}{B}_2'\,\sigma_B^2 + \underset{\sim}{I}_{rs}\,\sigma_{AB}^2 + \underset{\sim}{K}\,\sigma^2.$$

Lemma 1: $\underset{\sim}{B}_1\underset{\sim}{B}_1'$ and $\underset{\sim}{B}_2\underset{\sim}{B}_2'$ commute.

Proof: Multiplying $\underset{\sim}{B}_1\underset{\sim}{B}_1'$ with $\underset{\sim}{B}_2\underset{\sim}{B}_2'$, we get a symmetric matrix. When the product of two symmetric matrices is symmetric, the matrices commute. □

From lemma 1 it follows that there exists an orthogonal matrix $\mathbf{P}$ with the property that $\mathbf{P}\,\mathbf{A}_1\,\mathbf{P}'$ and $\mathbf{P}\,\mathbf{A}_2\,\mathbf{P}'$ are diagonal matrices with the eigenvalues on the diagonal (Herbach,1959). $\mathbf{P}$ may be chosen so that the first row, in $\mathbf{P}$ is $(rs)^{-\frac{1}{2}}(1,1,\ldots,1)$. ($\mathbf{A}_1 = \mathbf{B}_1\,\mathbf{B}_1'$; $\mathbf{A}_2 = \mathbf{B}_2\,\mathbf{B}_2'$).

If $\mathbf{Z} = \mathbf{P}\,\bar{\mathbf{y}}$, the covariance matrix for $\mathbf{Z}$ is

$$\boldsymbol{\Sigma}\,(\mathbf{Z}) = \mathbf{P}\,\mathbf{A}_1\,\mathbf{P}'\,\sigma_A^2 + \mathbf{P}\,\mathbf{A}_2\,\mathbf{P}' \cdot \sigma_B^2 + \mathbf{I}_{rs}\,\sigma_{AB}^2 + \mathbf{P}\,\mathbf{K}\,\mathbf{P}'\,\sigma^2.$$

<u>Lemma 2</u>: (i) Rank $(\mathbf{B}_1) = r$;

(ii) Rank $(\mathbf{B}_2) = s$;

(iii) Rank $(\mathbf{B}_1 \vdots \mathbf{B}_2) = r + s - 1$;

(iv) Rank $(\mathbf{A}_1 + \mathbf{A}_2) = $ rank $(\mathbf{B}_1 \vdots \mathbf{B}_2)$.

<u>Proof</u>: (i), (ii), and (iii) are seen from (2.4). (iv) follows from the proof of Graybill and Hultquist's (1961) theorem 1. $\square$

From the fact that rank $(\mathbf{A}_1) = $ rank $(\mathbf{B}_1) = r$ and because $\mathbf{A}_1$ has the eigenvalues s of multiplicity r and 0 of multiplicity $(r \cdot s - r) = r(s-1)$, it follows that $\mathbf{P}\,\mathbf{A}_1\,\mathbf{P}'$ has r diagonal elements all equal to s and the rest equal to 0. In the same way it is seen that $P\,A_2\,P'$ has s diagonal elements all equal to r and the other elements equal to 0.

From (iii) and (iv) it is seen that the matrix $(\mathbf{P}\,\mathbf{A}_1\,\mathbf{P}' + \mathbf{P}\,\mathbf{A}_2\mathbf{P}')$ has $(r + s - 1)$ diagonal elements different from zero. Thus when the diagonal element in $\mathbf{P}\,\mathbf{A}_1\,\mathbf{P}'$ is different from zero, the corresponding element in $\mathbf{P}\,\mathbf{A}_2\,\mathbf{P}'$ is equal to zero except in one place (in the first row).

We now partition $\mathbf{Z}$ in the following way:

(i) $Z_1 = (rs)^{\frac{1}{2}}\,\bar{y}\ldots$, which is the first element in $\mathbf{Z}$.

(ii) $\mathbf{Z}_A$ consists of the $(r - 1)$ elements in $\mathbf{Z}$ whose covariance matrix is independent of σ_B^2.

(iii) $\mathbf{Z}_B$ consists of the $(s - 1)$ elements in $\mathbf{Z}$ whose covariance matrix is independent of σ_A^2.

(iv) $\mathbf{Z}_{AB}$ consists of the $(r - 1)(s - 1)$ elements in $\mathbf{Z}$ whose covariance matrix is independent of σ_A^2 and σ_B^2.

Lemma 3: $E\underset{\sim}{Z}_A = E\underset{\sim}{Z}_B = E\underset{\sim}{Z}_{AB} = 0.$

Proof: This follows from the fact that $\underset{\sim}{P}$ is orthogonal with a first row which is $(rs)^{-\frac{1}{2}}(1,\ldots,1)$. $\square$

We have

$$\underset{\sim}{\Sigma}\,(\underset{\sim}{Z}_A) = s\,\underset{\sim}{I}_{r-1}\,\sigma_A^2 + \underset{\sim}{I}_{r-1}\,\sigma_{AB}^2 + \underset{\sim}{K}_1\,\sigma^2,$$

$$\underset{\sim}{\Sigma}\,(\underset{\sim}{Z}_B) = r\,\underset{\sim}{I}_{s-1}\,\sigma_B^2 + \underset{\sim}{I}_{s-1}\,\sigma_{AB}^2 + \underset{\sim}{K}_2\,\sigma^2,$$

and

$$\underset{\sim}{\Sigma}\,(\underset{\sim}{Z}_{AB}) = \underset{\sim}{I}_{(r-1)(s-1)}\,\sigma_{AB}^2 + \underset{\sim}{K}_3\,\sigma^2.$$

Here $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$ and $\underset{\sim}{K}_3$ are the corresponding submatrices of $\underset{\sim}{P}\,\underset{\sim}{K}\,\underset{\sim}{P}'$.

In what follows, $\underset{\sim}{Z}_A$, $\underset{\sim}{Z}_B$ and $\underset{\sim}{Z}_{AB}$ will be used for testing hypotheses concerning σ_A^2/σ^2, σ_B^2/σ^2, and σ_{AB}^2/σ^2.

2.a Test for σ_{AB}^2/σ^2

$\underset{\sim}{\Sigma}\,(\underset{\sim}{Z}_{AB})$ may be written as $(\underset{\sim}{I}_{(r-1)(s-1)}\,\Delta_{AB} + \underset{\sim}{K}_3)\sigma^2$, where $\Delta_{AB} = \sigma_{AB}^2/\sigma^2$. Then

$$Q_{AB} = \underset{\sim}{Z}'_{AB}\,(\underset{\sim}{I}_{(r-1)(s-1)}\,\Delta_{AB} + \underset{\sim}{K}_3)^{-1}\,\underset{\sim}{Z}_{AB}/\sigma^2 \tag{2.4}$$

has a X^2-distribution with $(r-1)(s-1)$ degrees of freedom. There exists an orthogonal matrix $\underset{\sim}{A}$ such that $\underset{\sim}{A}\,\underset{\sim}{K}_S\,\underset{\sim}{A}' = \underset{\sim}{D}_1$ is a diagonal matrix. Introduce $\underset{\sim}{Z}^*_{AB} = \underset{\sim}{A}\,\underset{\sim}{Z}_{AB}$. The covariance matrix for $\underset{\sim}{Z}^*_{AB}$ is $(\underset{\sim}{I}_{(r-1)(s-1)}\,\Delta_{AB} + \underset{\sim}{D}_1)$ and

$$\underset{\sim}{Z}'_{AB}(\underset{\sim}{I}_{(r-1)(s-1)}\,\Delta_{AB} + \underset{\sim}{K}_3)^{-1}\,\underset{\sim}{Z}_{AB} = \underset{\sim}{Z}^*_{AB}(\underset{\sim}{I}_{(r-1)(s-1)}\,\Delta_{AB} + \underset{\sim}{D}_1)^{-1}\,\underset{\sim}{Z}^*_{AB}$$

$$= \sum_{j=1}^{(r-1)(s-1)} (Z^*_{jAB})^2/(\Delta_{AB} + d_j).$$

Here $d_1,\ldots,d_{(r-1)(s-1)}$ are the diagonal elements of $\underset{\sim}{D}_1$. We see that Q_{AB} is a decreasing function of Δ_{AB}.

Define $Q = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij.})^2$. Then Q/σ^2 has a X^2-distribution with $(n-rs)$ degrees of freedom. Q is stochastically independent of Q_{AB}. Thus $F(\Delta_{AB}) = (n-rs)\,Q_{AB}/(r-1)(s-1)\,Q$ has an F-distribution. Since Q_{AB} decreases with Δ_{AB}, $F(\Delta_{AB})$ decreases with Δ_{AB}. Hence a confidence interval can be obtained in the usual way.

When testing the hypothesis

$$\Delta_{AB} \leq \Delta_0 \text{ against } \Delta_{AB} > \Delta_0,$$

we reject when $F(\Delta_C)$ is larger than the upper α-quantile, $f_{1-\alpha}$, of the corresponding F-distribution. The power function is

$$\beta(\Delta_{AB}) = P\{(n-rs)\left[\sum_{i=1}^{n} Z_{iAB}^2/(\Delta_0 + d_i)\right] / \left[(r-1)(s-1)\ Q)\right] > f_{1-\alpha}\}$$

$$= P\{(n-rs)\left[\sum_{i=1}^{n} (\Delta_{AB} + d_i)\ R_i/(\Delta_0 + d_i)\right] / \left[(r-1)(s-1)\right] > f_{1-\alpha}\}$$

where $R_1, \ldots, R_{(r-1)(s-1)}$ are independent X^2-distributed random variables with 1 degree of freedom. $\beta(\Delta_{AB})$ decreases with Δ_{AB}.

2.b. Test for σ_A^2/σ^2 assuming $\sigma_{AB} = 0$

When $\sigma_{AB} = 0$ the covariance matrix for $\begin{Bmatrix} \underset{\sim}{Z}_A \\ \underset{\sim}{Z}_{AB} \end{Bmatrix}$ is equal to

$$\underset{\sim}{\Sigma}\begin{Bmatrix} \underset{\sim}{Z}_A \\ \underset{\sim}{Z}_{AB} \end{Bmatrix} = \begin{Bmatrix} s\ \underset{\sim}{I}(r-1) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{Bmatrix} \sigma_A^2 + \begin{Bmatrix} \underset{\sim}{K}_1 & \underset{\sim}{K}_4 \\ \underset{\sim}{K}_4' & \underset{\sim}{K}_3 \end{Bmatrix} \sigma^2,$$

where $E\{\underset{\sim}{Z}_A\ \underset{\sim}{Z}_{AB}'\} = \underset{\sim}{K}_4$. $\begin{Bmatrix} \underset{\sim}{I}(r-1) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{Bmatrix}$ is positive semi-definite, and $\begin{Bmatrix} \underset{\sim}{K}_1 & \underset{\sim}{K}_4 \\ \underset{\sim}{K}_4 & \underset{\sim}{K}_3 \end{Bmatrix}$ is positive definite, so we can find a non-singular matrix H such that $\underset{\sim}{H}\begin{Bmatrix} \underset{\sim}{K}_1 & \underset{\sim}{K}_4 \\ \underset{\sim}{K}_4 & \underset{\sim}{K}_3 \end{Bmatrix}\underset{\sim}{H}' = \underset{\sim}{I}$, and $\underset{\sim}{H}\begin{Bmatrix} s\ \underset{\sim}{I}(r-1) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{Bmatrix}\underset{\sim}{H}' = \lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{r-1}, 0, \ldots, 0\}$.

Define $\underset{\sim}{U} = \begin{Bmatrix} \underset{\sim}{U}_A \\ \underset{\sim}{U}_{AB} \end{Bmatrix} = \underset{\sim}{H}\begin{Bmatrix} \underset{\sim}{Z}_A \\ \underset{\sim}{Z}_{AB} \end{Bmatrix}$. If $\Delta_A = \sigma_A^2/\sigma^2$, $Q_A = \underset{\sim}{U}_A'(\underset{\sim}{\lambda}\Delta_A + \underset{\sim}{I}_{(r-1)})^{-1}\ \underset{\sim}{U}_A/\sigma^2$ has a X^2-distribution with $(r-1)$ degrees of freedom, and $Q_{AB}^* = \underset{\sim}{U}_{AB}'\ \underset{\sim}{I}_{(r-1)(s-1)}\ \underset{\sim}{U}_{AB}/\sigma^2$ has a X^2-distribution with $(r-1)(s-1)$ degrees of freedom. Q_A, Q_{AB}^* and Q are stochastically independent.

To test the hypothesis $\Delta_A \leq \Delta_0$ against $\Delta_A > \Delta_0$, we reject when

$$G(\Delta_A) = Q_A\{(n-rs) + (r-1)(s-1)\}/(Q + Q_{AB})(r-1) \tag{2.5}$$

is larger than the upper α-quantile, $f_{1-\alpha}$, of the corresponding F-distribution.

In the same way as above it may be proved that the test is unbiased. A similar test exists concerning σ_B^2/σ^2.

3. On the possibility of testing hypotheses concerning σ_A^2/σ^2 without assuming $\sigma_{AB} = 0$

In balanced experimental design models we know that

$$(r-1)(s-1)\mathbf{Z}_A'(s\mathbf{I}_{(r-1)}\sigma_A^2+\mathbf{I}_{(r-1)}\sigma_{AB}^2+\mathbf{K}_1\sigma^2)^{-1}\ \mathbf{Z}_A/(r-1)\ \mathbf{Z}_{AB}'(\mathbf{I}_{(r-1)(s-1)}\sigma_{AB}^2+\mathbf{K}_3\sigma^2)^{-1}\ \mathbf{Z}_{AB}$$

(3.1)

$$= (r-1)(s-1)\mathbf{Z}_A'(s\mathbf{I}_{(r-1)}\sigma_A^2 + \mathbf{I}_{(s-1)}\Delta_{AB} + K_1)^{-1}\ \mathbf{Z}_A/(r-1)\ \mathbf{Z}_{AB}'(\mathbf{I}_{(r-1)}\Delta_{AB}+\mathbf{K}_3)^{-1}\ \mathbf{Z}_{AB}$$

is F-distributed. This is not always the case in unbalanced models because $\mathbf{Z}_A$ and $\mathbf{Z}_{AB}$ may not be stochastically independent. Let us now assume that $\mathbf{Z}_A$ and $\mathbf{Z}_{AB}$ are stochastically independent (this may happen even in an unbalanced model). Define two orthogonal matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ such that $\mathbf{M}_1\ \mathbf{K}_1\ \mathbf{M}_1' = \mathbf{L}_1$ and $\mathbf{M}_2\ \mathbf{K}_3\ \mathbf{M}_2' = \mathbf{L}_2$ are diagonal. Let $\mathbf{V}_A = \mathbf{M}_1\ \mathbf{Z}_A$ and $\mathbf{V}_{AB} = \mathbf{M}_2\ \mathbf{Z}_{AB}$. Then (3.1) may be written as

$$(3.2)\qquad (r-1)(s-1)\left[\sum_{i=1}^{r-1} V_{iA}^2/(s\Delta_A + \Delta_{AB} + \ell_{1i})\right] \Big/ \left[(r-1\ \sum_{j=1}^{(r-1)(s-1)} V_{jAB}^2/)\Delta_{AB} + \ell_2)\right]$$

where ℓ_{1i} and ℓ_{2i} are the diagonal elements of $\mathbf{L}_1$ and $\mathbf{L}_2$. The quantity in (3.2) has an F-distribution, but the assumption that $\mathbf{Z}_A$ and $\mathbf{Z}_{AB}$ are stochastically independent is not sufficient to give a test for the hypothesis $\Delta_A \leq \Delta_0$ against $\Delta_A > \Delta_0$.

In cases where

(3.3) $\ell_{1i} = \ell_{2j} = \ell$ for all i and j, formula (3.2) is reduced to

$$(\Delta_{AB} + \ell)(r-1)(s-1)\sum_{i=1}^{r-1} V_{iA}^2/(r-1)(s\Delta_A + \Delta_{AB} + \ell)\sum_{j=1}^{(r-1)(s-1)} V_{jAB}^2.$$

If the null hypothesis is $\Delta_A = 0$, we have that $g(\Delta_A) = (s-1)(r-1)\sum_{i=1}^{r-1} Vi_A^2/(r-1)\sum_{j=1}^{(r-1)(s-1)} V_{jAB}^2$ is F-distributed under the null hypothesis. Hence we reject if g(0) is larger than the upper α-quantile of the corresponding F-distribution.

In the case r = s = 2 assumption (3.2) is always fullfilled.

4. Comparison with corresponding tests in fixed effects models

A two-way layout in fixed effects models may be described as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk};$$

$i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$; $k = 1,2,\ldots,n_{ij}$, where μ, α_i, β_j, and γ_{ij} are unknown constants such that

$$(4.1) \qquad \sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0,$$

and the e_{ijk} have a joint normal distribution with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{I}_n \sigma^2$.

The null hypothesis $\gamma_{ij} = 0$ ($i = 1,2,\ldots,r$; $j = = 1,2,\ldots,s$) is tested by minimizing the sum of squares $Q = \sum_{i,j,k} (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2$ under the null hypothesis and under the a priori specifications. Let the two minima of Q be Q_ω and Q_Ω, respectively. The null hypothesis is rejected when

$$(4.2) \qquad (Q_\omega - Q_\Omega)(n-rs)/Q_\Omega(r-1)(s-1)$$

is larger than the upper α-quantile $f_{1-\alpha}$ of the corresponding F-distribution.

We will prove that the quantity in (4.2) is equal to the test-statistic F(0) in section 2a.

If as in section 2 we introduce $\bar{\underset{\approx}{y}}$ we have that

$$(4.3) \qquad \bar{\underset{\approx}{y}} = \underset{\sim}{J}_{rs}\, \mu + \underset{\sim}{B}_1 \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{Bmatrix} + \underset{\sim}{B}_2 \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_s \end{Bmatrix} + \underset{\sim}{I}_{rs}\, \underset{\approx}{\gamma} + \bar{\underset{\approx}{e}}.$$

The only difference from the random effects model (2.4) is that α_i, β_j, and γ_{ij} here are fixed constants with the side conditions (4.1). We write the side conditions in the form

$$\alpha_r = - \sum_{i=1}^{r-1} \alpha_i; \quad \beta_s = - \sum_{j=i}^{s-1} \beta_j;$$

$$\gamma_{is} = - \sum_{j=i}^{s-1} \gamma_{ij}; \quad \gamma_{rj} = - \sum_{i=1}^{r-1} \gamma_{ij};$$

and $$\gamma_{rs} = \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \gamma_{ij}.$$

The (4.3) takes the form

$$(4.5) \qquad \bar{\underset{\approx}{y}} = \underset{\sim}{Z} \begin{Bmatrix} \mu \\ \underset{\sim}{\alpha}^{*} \\ \underset{\sim}{\beta}^{*} \\ \underset{\sim}{\gamma}^{*} \end{Bmatrix} + \bar{\underset{\approx}{e}},$$

where $\underset{\sim}{\alpha}^{*} = (\alpha_1,\ldots,\alpha_{r-1})'$; $\underset{\sim}{\beta}^{*} = (\beta_1,\ldots,\beta_{s-1})'$; $\underset{\sim}{\gamma}^{*} = (\gamma_1,\ldots,\gamma_{(r-1)(s-1)})'$; $\underset{\sim}{Z}$ is a quadratic, non-singular $(rs \times rs)$-matrix and $\bar{\underset{\sim}{e}}$ is normally distributed with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{K}\sigma^2$, with $\underset{\sim}{K}$ given as above (2.3). (It is possible to write (4.1) in several other ways. This will lead to formally different $\underset{\sim}{Z}$ matrices, and formally different $\underset{\sim}{\alpha}^{*}$, $\underset{\sim}{\beta}^{*}$ and $\underset{\sim}{\gamma}^{*}$ in (4.5)). Define $\underset{\sim}{V} = \underset{\sim}{K}^{-\frac{1}{2}} \bar{\underset{\sim}{Y}}$. Then

$$(4.6) \qquad \underset{\sim}{V} = \underset{\sim}{K}^{-\frac{1}{2}}\underset{\sim}{Z} \begin{bmatrix} \mu \\ \underset{\sim}{\alpha}^{*} \\ \underset{\sim}{\beta}^{*} \\ \underset{\sim}{\gamma}^{*} \end{bmatrix} + \underset{\sim}{e}^{*},$$

where $\underset{\sim}{e}^{*}$ is normally distributed with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{I}_{rs}\sigma^2$.

The form (4.6) is very convenient because to minimize Q is equivalent to minimize $(\underset{\sim}{V} - E\underset{\sim}{V})'(\underset{\sim}{V} - E\underset{\sim}{V})$. This is seen as follows: With the side conditions (4.4) on the parameters, Q may be written

$$Q = \sum_{i,j,k} (y_{ijk} - y_{ij.})^2 + \sum_{i=1}^{r-1}\sum_{j=1}^{s-1} n_{ij}(y_{ij} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 +$$

$$(4.7) \qquad \sum_{j=1}^{s-1} n_{rj}(y_{rj.} - \mu + \sum_{i=1}^{r-1}\alpha_i - \beta_j + \sum_{i=1}^{r-1}\gamma_{ij})^2 +$$

$$\sum_{i=1}^{r-1} n_{is}(y_{is} - \mu - \alpha_i + \sum_{j=1}^{s-1}\beta_j + \sum_{j=1}^{s-1}\gamma_{ij})^2 +$$

$$n_{rs}(y_{rs.} - \mu + \sum_{i=1}^{r-1}\alpha_i + \sum_{j=1}^{s-1}\beta_i + \sum_{j=1}^{s-1}\beta_j - \sum_{i=1}^{r-1}\sum_{j=1}^{s-1}\gamma_{ij})^2$$

The part of Q which depends on the parameters, equals

$$(4.8) \qquad Q_p = (\underset{\sim}{V} - E\underset{\sim}{V})'(\underset{\sim}{V} - E\underset{\sim}{V}).$$

The minimum of Q is then equal to the minimum of Q_p plus $\sum_{i,j,k}(y_{ijk} - y_{ij.})^2$. Define $Q_{p\Omega}$ and $Q_{p\omega}$ as the minima of Q_p under the a priori specifications and under the null hypothesis, respectively. We then have

<u>Lemma 4</u>: $Q_\omega - Q_\Omega = Q_{p\omega} - Q_{p\Omega}$.

The a priori specifications are (4.4), and the null hypothesis is

$\gamma_{ij} = 0 \; (i = 1,2,\ldots,r-1;\; j = 1,2,\ldots,s-1)$

From the general theory for linear models we know that

$$(4.9) \qquad Q_{p\omega} - Q_{p\Omega} = \hat{\gamma}^{*\prime}(\Sigma_4)^{-1}\,\hat{\gamma}^{*},$$

where $\hat{\gamma}^{*}$ is the least squares estimate for γ^{*}, and Σ_4 is the covariance matrix for $\hat{\gamma}^{*}$.

The least squares estimate for $\begin{bmatrix} \mu \\ \alpha^{*} \\ \beta^{*} \\ \gamma^{*} \end{bmatrix}$ is

$$\begin{bmatrix} \mu \\ \hat{\alpha}^{*} \\ \hat{\beta}^{*} \\ \hat{\gamma}^{*} \end{bmatrix} = (Z'\,K^{-\frac{1}{2}}\,K^{-\frac{1}{2}}\,Z)^{-1}\,Z\,K^{-\frac{1}{2}}\,V,$$

which reduces to

$$\begin{bmatrix} \hat{\mu} \\ \hat{\alpha}^{*} \\ \hat{\beta}^{*} \\ \hat{\gamma}^{*} \end{bmatrix} = Z^{-1}\,\bar{y}.$$

The covariance matrix for this estimator is $\Sigma = (Z'\,K\,Z)^{-1}\,\sigma^2$.

By introducing the transformation P, where P is the orthogonal matrix with which the cell mean values were transformed in the corresponding random effect model, we will now prove that $Q_{p\omega} - Q_{p\Omega}$ is independent of the choice of Z, α^{*}, β^{*}, and γ^{*} and that $\sigma^{-2}(Q_{p\omega}-Q_{p\Omega}) = Q_{AB}$ when $\Delta_{AB} = 0$, where Q_{AB} is defined as in section 2.

The following lemma is usefull:

<u>Lemma 5</u>: Partition Z into submatrices corresponding to the partitioning $(\hat{\mu}, \hat{\alpha}^{*}, \hat{\beta}^{*}, \hat{\gamma}^{*})'$. Thus

$$Z = \left[J_{rs},\ Z^{(rs \times (r-1))},\ Z_2^{(rs \times (s-1))},\ Z_3^{(rs \times (r-1)(s-1)} \right].$$

Partition P likewise into

$$P = \begin{bmatrix} P_1^{(1\times rs)}{}_{((r-1)\times rs)} \\ P_{2((s-1)\times rs)} \\ P_{3((s-1)(r-1)\times rs} \\ P_4 \end{bmatrix}.$$

For any choice of $\underset{\sim}{Z}$ we then have:

(i) The rows of $\underset{\sim}{P}_2$ are orthogonal to the columns in $\underset{\sim}{Z}_2$.

(ii) The rows of $\underset{\sim}{P}_3$ are orthogonal to the columns in $\underset{\sim}{Z}_1$.

(iii) The rows of $\underset{\sim}{P}_4$ are orthogonal to the columns in $\underset{\sim}{Z}_1$ and $\underset{\sim}{Z}_2$.

Proof: By section 2 we can find a matrix $\underset{\sim}{P}$ such that $\underset{\sim}{P}\,\underset{\sim}{A}_1\,\underset{\sim}{P}' = \begin{bmatrix} s\underset{\sim}{I}_s & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}$ and $\underset{\sim}{P}\,\underset{\sim}{A}_2\,\underset{\sim}{P}' = \begin{bmatrix} r\underset{\sim}{I}_r & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}$. By the partitioning of $\underset{\sim}{P}$ introduced in the proof of lemma 3, $\underset{\sim}{P}_1\,\underset{\sim}{B}_1\,\underset{\sim}{B}_1'\,\underset{\sim}{P}_1' = s$, $\underset{\sim}{P}_1\,\underset{\sim}{B}_2\,\underset{\sim}{B}_2'\,\underset{\sim}{P}_1' = r$, $\underset{\sim}{P}_2\,\underset{\sim}{B}_1\,\underset{\sim}{B}_1'\,\underset{\sim}{P}_2' = s\,\underset{\sim}{I}_{(r-1)(r-1)}$, $\underset{\sim}{P}_2\,\underset{\sim}{B}_2\,\underset{\sim}{B}_2'\,\underset{\sim}{P}_2' = \underset{\sim}{0}_{(r-1)(r-1)}$, $\underset{\sim}{P}_3\,\underset{\sim}{B}_1\,\underset{\sim}{B}_1'\,\underset{\sim}{P}_3' = \underset{\sim}{0}_{(s-1)(s-1)}$, $\underset{\sim}{P}_3\,\underset{\sim}{B}_2\,\underset{\sim}{B}_2'\,\underset{\sim}{P}_3' = r\underset{\sim}{I}_{(s-1)(s-1)}$, $\underset{\sim}{P}_4\,\underset{\sim}{B}_1\,\underset{\sim}{B}_1'\,\underset{\sim}{P}_4' = \underset{\sim}{0}_{(r-1)(s-1)\times(r-1)(s-1)}$, and $\underset{\sim}{P}_4\,\underset{\sim}{B}_2\,\underset{\sim}{B}_2'\,\underset{\sim}{P}_3' = \underset{\sim}{0}_{(r-1)(s-1)\times(r-1)(s-1)}$.

It is always possible to find matrices A, B, C such that

$$\underset{\sim}{\alpha}^{r\times 1} = \underset{\sim}{A}^{(r\times(r-1))}\,\underset{\sim}{\alpha}^{*((r-1)\times 1)},$$
$$\underset{\sim}{\beta}^{s\times 1} = \underset{\sim}{B}^{(s\times(s-1))}\,\underset{\sim}{\beta}^{*((s-1)\times 1)}$$
$$\underset{\sim}{\gamma}^{(rs\times 1)} = \underset{\sim}{C}^{(rs\times(r-1)(s-1)}\,\underset{\sim}{\gamma}^{*(r-1)(s-1)\times 1)}.$$

Formula (2.4) may now be written

$$\underset{\sim}{\bar{Y}} = \underset{\sim}{\gamma}_{rs}^{(rs\times 1)}\,\mu + \underset{\sim}{B}_1\,\underset{\sim}{A}\,\underset{\sim}{\alpha}^* + \underset{\sim}{B}_2\,\underset{\sim}{B}\,\underset{\sim}{\beta}^* + \underset{\sim}{C}\,\underset{\sim}{\gamma}^* + \underset{\sim}{\bar{e}}$$

$\underset{\sim}{B}_1\,\underset{\sim}{A}$ and $\underset{\sim}{B}_2\,\underset{\sim}{B}$ equal $\underset{\sim}{Z}$, and $\underset{\sim}{Z}_2$ in lemma 5, respectively, and $\underset{\sim}{C}$ equals $\underset{\sim}{Z}_3$. The columns in $\underset{\sim}{B}_1\,\underset{\sim}{A}$ are linear combinations of the columns in $\underset{\sim}{B}_1$, so that $\mathcal{C}(\underset{\sim}{B}_1\,\underset{\sim}{A}) \subset \mathcal{C}(\underset{\sim}{B}_1)$, where $\mathcal{C}(\underset{\sim}{U})$ denotes the vector space spanned by the columns in any matrix $\underset{\sim}{U}$.

Thus $\mathcal{C}(\underset{\sim}{Z}_1) \subset \mathcal{C}(\underset{\sim}{B}_1)$ and $\mathcal{C}(\underset{\sim}{Z}_2) \subset \mathcal{C}(\underset{\sim}{B}_2)$. Then since $\underset{\sim}{P}_2\underset{\sim}{B}_2\underset{\sim}{B}_2'\underset{\sim}{P}_2' = \underset{\sim}{0}$, $\underset{\sim}{P}_2\underset{\sim}{B}_2 = \underset{\sim}{0}$ and thus $\underset{\sim}{P}_2\,\underset{\sim}{Z}_2 = \underset{\sim}{0}$, so the rows in $\underset{\sim}{P}_2$ are orthogonal to the columns in $\underset{\sim}{Z}_2$. The rest of the lemma now follows by treating $\underset{\sim}{P}_3$ and $\underset{\sim}{P}_4$ in a similar way. □

Because $\underset{\sim}{P}_2\,\underset{\sim}{J}_{rs} = \underset{\sim}{P}_3\,\underset{\sim}{J}_{rs} = \underset{\sim}{P}_4\,\underset{\sim}{J}_{rs} = \underset{\sim}{0}$, it follows by lemma 5 that $\underset{\sim}{P}\underset{\sim}{Z}$ has the form

$$PZ = \begin{bmatrix} \underset{\sim}{P}_1\,\underset{\sim}{J}_{rs} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}_2\,\underset{\sim}{Z}_1 & \underset{\sim}{0} & \underset{\sim}{P}_2\,\underset{\sim}{Z}_3 \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{P}_3\,\underset{\sim}{Z}_2 & \underset{\sim}{P}_3\,\underset{\sim}{Z}_3 \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{P}_4\,\underset{\sim}{Z}_3 \end{bmatrix}.$$

We then see that $(P\ Z)^{-1}$ also is a triangular matrix with zeroes to the left of the diagonal. The $(r-1)(s-1) \times (r-1)(s-1)$ submatrix in the lower, right hand corner of $(P\ Z)^{-1}$ equals $(P_4\ Z_3)^{-1}$.

Introduce $\underset{\sim}{P}$ into the expression for the least squares estimate and its covariance matrix, we obtain:

$$\begin{bmatrix} \hat{\underset{\sim}{\mu}} \\ \hat{\underset{\sim}{\alpha}}^* \\ \hat{\underset{\sim}{\beta}}^* \\ \hat{\underset{\sim}{\gamma}}^* \end{bmatrix} = \underset{\sim}{Z}^{-1}\ \bar{\underset{\sim}{Y}} = (\underset{\sim}{P}\ \underset{\sim}{Z})^{-1}\ \underset{\sim}{P}\ \bar{\underset{\sim}{Y}}$$

and $\underset{\sim}{\Sigma} = (\underset{\sim}{Z}'\ \underset{\sim}{K}^{-1}\ \underset{\sim}{Z})^{-1}\sigma^2 = (\underset{\sim}{P}\ \underset{\sim}{Z})^{-1}\ \underset{\sim}{P}\ \underset{\sim}{K}\ \underset{\sim}{P}'\ (\underset{\sim}{P}\ \underset{\sim}{Z})'^{-1}\sigma^2$. From what we found about $(\underset{\sim}{P}\ \underset{\sim}{Z})^{-1}$, it follows that the $(r-1)(s-1)$ lower elements of $(\underset{\sim}{P}\ \underset{\sim}{Z})^{-1}\ \underset{\sim}{P}\ \bar{\underset{\sim}{Y}}$ are $\hat{\underset{\sim}{\gamma}}^* = (\underset{\sim}{P}_4\ \underset{\sim}{Z}_3)^{-1}\ \underset{\sim}{P}_4\ \bar{\underset{\sim}{Y}}$, and the corresponding part of the covariance matrix is $(\underset{\sim}{P}_4\ \underset{\sim}{Z}_3)^{-1}\ (\underset{\sim}{P}\ \underset{\sim}{K}\ \underset{\sim}{P}')_4\ (\underset{\sim}{P}_4\ \underset{\sim}{Z}_3)^{-1}$, where $(\underset{\sim}{P}\ \underset{\sim}{K}\ \underset{\sim}{P}')_4$ is the $((r-1)(s-1) + (r-1)(s-1)$ submatrix in the lower right hand corner og $P\ K\ P'$. (4.9) may then be written in the form

$$\bar{\underset{\sim}{Y}}'\ \underset{\sim}{P}_4'\ (\underset{\sim}{P}_4\ \underset{\sim}{Z}_3)'^{-1}\ (\underset{\sim}{P}_4\ \underset{\sim}{Z}_3)'\ (\underset{\sim}{P}\ \underset{\sim}{K}\ \underset{\sim}{P}')_4^{-1}\ (\underset{\sim}{P}_4\ \underset{\sim}{Z}_4)(\underset{\sim}{P}_4\ \underset{\sim}{Z}_4)^{-1}\ \underset{\sim}{P}_4\ \bar{\underset{\sim}{Y}}\ \sigma^2$$

$$(4.10) \qquad = \bar{\underset{\sim}{Y}}'\ \underset{\sim}{P}_4'\ (\underset{\sim}{P}\ \underset{\sim}{K}\ \underset{\sim}{P}')_4^{-1}\ \underset{\sim}{P}_4\ \bar{\underset{\sim}{Y}}\ \sigma^2.$$

This quadratic form is independent of $\underset{\sim}{Z}_1\ \underset{\sim}{\alpha}^*$, $\underset{\sim}{\beta}^*$ and $\underset{\sim}{\gamma}^*$, and is the same as Q_{AB} in (2.4) when $\Delta_{AB} = 0$, because $\underset{\sim}{Z}_{AB} = \underset{\sim}{P}_4\ \bar{\underset{\sim}{Y}}$ and $\underset{\sim}{K}_3 = (\underset{\sim}{P}\ \underset{\sim}{K}\ \underset{\sim}{P}')_4$. We have then proved that $(n-rs)(Q_\omega - Q_\Omega)/Q_\Omega(r-1)(s-1) = F(0)$.

5. The test statistics expressed by the original observations

Lemma 6: With the choice of $\underset{\sim}{Z}$ made in section 4, the least squares estimates for $(\mu, \underset{\sim}{\alpha}^*, \underset{\sim}{\beta}^*, \underset{\sim}{\gamma}^*)'$ are $\hat{\mu} = y_{...}$, $\{\hat{\alpha}_i^*\} = \{y_{i..} - y_{...}\}$, $\{\hat{\beta}^*\} = \{y_{.j.} - y_{...}\}$, and $\{\hat{\gamma}_{ij}^*\} = \{y_{ij.} - y_{i..} - y_{.j.} + y_{...}\}$. $(i = 1,2,\ldots,r-1;\ j = 1,2,\ldots,s-1)$.

Proof: If we insert $\hat{\mu}$, $\{\hat{\alpha}_i^*\}$, $\{\hat{\beta}_j^*\}$ and $\{\hat{\gamma}_{ij}^*\}$ for μ, $\{\alpha_i\}$, $\{\beta_j\}$ and $\{\gamma_{ij}\}$ in (4.7), Q reduces to $\sum_{i,j,k} (y_{ijk} - y_{ij.})^2$. □

When testing the null hypothesis $\Delta_{AB} \leq 0$ against $\Delta_{AB} > 0$, we reject when

$$(5.1) \qquad (n-rs)\ \hat{\gamma}^{*\prime}\ (\underset{\sim}{\Sigma}_4)^{-1}\ \hat{\gamma}^* / \sum_{i,j,k} (y_{ijk} - y_{ij.})^2\ (r-1)(s-1)$$

is larger than the upper -quantile of the corresponding F-distribution. This test is the same as the one suggested by Spjøtvoll (1968).

It should be noted that the test statistic reduces to the usual one when the model is balanved.

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